Construction of doubly-periodic instantons

Marcos Jardim
Yale University
Department of Mathematics
10 Hillhouse Avenue
New Haven, CT 06520-8283 USA

February 1, 2008

Abstract

We construct finite-energy instanton connections over $\mathbb{R}^4$ which are periodic in two directions via an analogue of the Nahm transform for certain singular solutions of Hitchin's equations defined over a 2-torus.

1 Introduction

Since the appearance of the Yang-Mills equation on the mathematical scene in the late 70's, its anti-self-dual (ASD) solutions have been intensively studied. The first major result in the field was the ADHM construction of instantons on $\mathbb{R}^4$ [1]. Soon after that, W. Nahm adapted the ADHM construction to obtain the *time-invariant* ASD solutions of the Yang-Mills equations, the so-called monopoles [18]. It turns out that these constructions are two examples of a much more general framework.

The *Nahm transform* can be defined in general for anti-self-dual connections on $\mathbb{R}^4$, which are invariant under some subgroup of translations $\Lambda \subset \mathbb{R}^4$ (see [19]). In these generalised situations, the Nahm transform gives rise to *dual instantons* on $(\mathbb{R}^4)^*$, which are invariant under:

$$\Lambda^* = \{\alpha \in (\mathbb{R}^4)^* \mid \alpha(\lambda) \in \mathbb{Z}\ \forall \lambda \in \Lambda\}$$

There are plenty of examples of such constructions available in the literature, namely:

- The trivial case $\Lambda = \{0\}$ is closely related to the celebrated ADHM construction of instantons, as described by Donaldson & Kronheimer [7]; in this case, $\Lambda^* = (\mathbb{R}^4)^*$ and an instanton on $\mathbb{R}^4$ corresponds to some algebraic data.

- If $\Lambda = \mathbb{Z}^4$, this is the Nahm transform of Braam & van Baal [5] and Donaldson & Kronheimer [7], defining a hyperkähler isometry of the moduli space of instantons over two dual 4-tori.

- $\Lambda = \mathbb{R}$ gives rise to monopoles, extensively studied by Hitchin [10], Donaldson [6], Hurtubise & Murray [12] and Nakajima [19], among several others; here, $\Lambda^* = \mathbb{R}^3$, and the transformed object is, for SU(2) monoples, an analytic solution of certain matrix-valued ODE's (the so-called Nahm's equations), defined over the open interval $(0, 2)$ and with simple poles at the end-points.

- $\Lambda = \mathbb{Z}$ correspond to the so-called calorons, studied by Nahm [18], Garland & Murray [8] and others; the transformed object is the solution of certain nonlinear Nahm-type equations on a circle.

The purpose of this paper fits well into this larger mathematical programme. Our goal is to construct finite-energy instantons over $\mathbb{R}^4$ provided with the Euclidean metric, which are periodic in two directions ($\Lambda^* = \mathbb{Z}^2$), so-called *doubly-periodic instantons*, from solutions of Hitchin's equations [11] defined on a 2-torus, i.e. instantons over $\mathbb{R}^4$ which are invariant under $\Lambda = \mathbb{Z}^2 \times \mathbb{R}^2$. The latter object is now very well studied, and their existence is determined by certain holomorphic data.

One might also ask if all doubly-periodic instantons can be produced in this way. In the sequel [14] of this paper, we will show that the construction here presented is invertible by describing the Nahm transform for instantons over $T^2 \times \mathbb{R}^2$, which produce singular solutions of Hitchin's equations.

Indeed, Hitchin's equations admit very few smooth solutions over elliptic curves (see [11]). Therefore, by analogy with Hitchin's construction of monopoles [10], we will consider a certain class of singular solutions, for which existence is guaranteed [16, 21]. The singularity data is converted into the asymptotic behaviour of the Nahm transformed doubly-periodic instanton; such a picture is again familiar from the construction of monopoles.

A string-theoretical version of the Nahm transform here presented was given by Kapustin & Sethi [15]. In fact, the other examples of Nahm transforms mentioned above also have string-theoretical interpretations. The ADHM construction and the Fourier transform of instantons over 4-tori were discussed in these terms by Witten [22], while Kapustin & Sethi [15] also treated the case of calorons.

Let us now outline the contents of this paper. Section 2 is dedicated to a brief review of Hitchin's self-duality equations, and the precise description of the particular type of solutions we will be interested in. The main topic of the paper is contained in sections 3 and 4, when we will show how to

construct doubly-periodic instantons and explore some of the properties of the instantons obtained. We conclude with a few remarks and raising some questions for future investigation.

Acknowledgements. This work is part of my Ph.D. project [13], which was funded by CNPq, Brazil. I am grateful to my supervisors, Simon Donaldson and Nigel Hitchin, for their constant support and guidance. I also thank Olivier Biquard, Alexei Kovalev and Brian Steer for invaluable help in the later stages of the project.

2 Singular Higgs pairs

In [11] Hitchin studied the dimensional reduction of the usual Yang-Mills anti-self-dual equations from four to two dimensions. More precisely, let $V \to \mathbb{R}^4$ be a rank k vector bundle with a connection $\tilde{B}$ which does not depend on two coordinates. Pick up a global trivialisation of V and write down $\tilde{B}$ as a 1-form:

$$\tilde{B} = B_1(x,y)dx + B_2(x,y)dy + \phi_1(x,y)dz + \phi_2(x,y)dw$$

Hitchin then defined a *Higgs field* $\Phi = (\phi_1 - i\phi_2)d\xi$, where $d\xi = dx + idy$. So Φ is a section of $\Lambda^{1,0}\mathrm{End}V$, where V is now seen as a bundle over $\mathbb{R}^2$ with a connection $B = B_1 dx + B_2 dy$.

The ASD equations for $\tilde{B}$ over $\mathbb{R}^4$ can then be rewritten as a pair of equations on (B, Φ) over $\mathbb{R}^2$:

$$\begin{cases} F_B + [\Phi, \Phi^*] = 0 \\ \overline{\partial}_B \Phi = 0 \end{cases} \tag{1}$$

These equations are also conformally invariant, so they make sense over any Riemann surface. Solutions (B, Φ) are often called *Higgs pairs*.

As we mentioned in the introduction, we are interested in singular Higgs pairs over a 2-torus $\hat{T}$ defined on an $U(k)$-bundle $V \to \hat{T}$. Since we want to

think of $\hat{T}$ as a quotient of $\mathbb{R}^4$ by $\Lambda = \mathbb{Z}^2 \times \mathbb{R}^2$, the natural choice of metric for $\hat{T}$ is the flat, Euclidean metric. Let us also fix a complex structure on $\hat{T}$ coming from a choice of complex structure on $\mathbb{R}^4$.

Singular Higgs bundles were widely studied by many authors ([21], [17] and [16] among others) and are closely related to the so-called *parabolic Higgs bundles*. Adopting this point of view, we will consider a holomorphic vector bundle $\mathcal{V} \to \hat{T}$ of degree -2 with the following quasi-parabolic structure over two points $\pm\xi_0 \in \hat{T}$ (regarding now $\hat{T}$ as an elliptic curve):

$$\mathcal{V}_{\pm\xi_0} = F_1\mathcal{V}_{\pm\xi_0} \supset \underbrace{F_2\mathcal{V}_{\pm\xi_0}}_{\dim = 1} \supset F_3\mathcal{V}_{\pm\xi_0} = \{0\} \qquad \text{order}(\xi_0) \neq 2$$

$$\mathcal{V}_{\xi_0} = F_1\mathcal{V}_{\xi_0} \supset \underbrace{F_2\mathcal{V}_{\xi_0}}_{\dim = 2} \supset \underbrace{F_3\mathcal{V}_{\xi_0}}_{\dim = 1} \supset F_4\mathcal{V}_{\xi_0} = \{0\} \qquad \text{order}(\xi_0) = 2$$

To complete the parabolic structure we need to assign *weights* $\alpha_1(\pm\xi_0)$ to $F_1\mathcal{V}_{\pm\xi_0}$ and $\alpha_2(\pm\xi_0)$ to $F_2\mathcal{V}_{\pm\xi_0}$ if $\xi_0 \neq -\xi_0$ or $\alpha_1(\xi_0)$ to $F_1\mathcal{V}_{\xi_0}$, $\alpha_2(\xi_0)$ to $F_2\mathcal{V}_{\xi_0}$ and $\alpha_3(\xi_0)$ to $F_3\mathcal{V}_{\xi_0}$ if $\xi_0 = -\xi_0$. We assume that $\alpha_1 = 0$ in both cases; if ξ_0 is not of order two, we fix that $\alpha_2(\xi_0) = 1 + \alpha$ and $\alpha_2(-\xi_0) = 1 - \alpha$; if ξ_0 has order two, we fix that $\alpha_2(\xi_0) = 1 - \alpha$ and $\alpha_3(\xi_0) = 1 + \alpha$ for some $0 \leq \alpha < \frac{1}{2}$. Note in particular that $\mathcal{V}$ with this parabolic structure has zero parabolic degree.

From the point of view of the Higgs pair (B, Φ), this means that the bundle V is defined away from $\pm\xi_0$, and satisfies, holomorphically:

$$\mathcal{V}|_{\hat{T}\setminus\{\pm\xi_0\}} \simeq (V, \overline{\partial}_B)$$

The Higgs field Φ has simple poles at the parabolic points $\pm\xi_0 \in \hat{T}$ such that the residues $\phi_0(\pm\xi_0)$ of Φ are $k \times k$ matrices of rank 1. If ξ_0 is one of the four elements of order 2 in $\hat{T}$, then the residue $\phi_0(\xi_0)$ is assumed to be a $k \times k$ matrix of rank 2.

Moreover, the harmonic metric h associated with the Higgs pair (B, Φ) is assumed to be compatible with the parabolic structure. This means that,

in a holomorphic trivialisation of V over a sufficiently small neighbourhood around $\pm\xi_0$, h is non-degenerate along the kernel of the residues of Φ, and $h \sim O(r^{1\pm\alpha})$ along the image of the residues of Φ.

Such metric is clearly not a hermitian metric on the extended bundle $\mathcal{V}$ (since it degenerates at $\pm\xi_0$). Let h' be a hermitian metric on $\mathcal{V}$ bounding above the harmonic metric on V.

If $(\mathcal{V}, \Phi)$ is α-stable in the sense of parabolic Higgs bundles, then the existence of a meromorphic Higgs pair as above is guaranteed [21] for any rank k and any choice of $\pm\xi_0$.

Moreover, one usually fixes the eigenvalues of the residues of Φ as well. In our situation, this amounts to choosing only one complex number that we denote by ϵ. We assume that $\epsilon \neq 0$, i.e. the residues of Φ are semi-simple.

However, in this paper, these parameters (the weights α_i and the eigenvalue of the residues ϵ) will be allowed to vary; see [4] for a complete discussion. It is reassuring to know that if two sets of parameters (α, ϵ) and (α', ϵ') are chosen in generic position, then α-stability and α'-stability are in fact equivalent conditions [20].

In particular, the case $k = 1$ is very simple: once the parameters parameters (α, ϵ) are fixed and for any choice of $\pm\xi_0$, the moduli space of meromorphic Higgs pairs is just the cotangent bundle of T, that is a copy of $T \times \mathbb{C}$.

We will study solutions of (1) over $\hat{T}$ with the singularities $\pm\xi_0$ removed. Due to the non-compactness of $\hat{T} \setminus \{\pm\xi_0\}$, the choice of metric on the base space is a delicate issue. From the point of view of the Nahm transform, it is important to consider the Euclidean, incomplete metric on the punctured torus, as it is well-known from the examples mentioned above. However, such a choice of metric is not a good one from the analytical point of view. For instance, one cannot expect, on general grounds, to have a finite dimensional moduli space of Higgs pairs.

Fortunately, as we mentioned before, Hitchin's equations are conformally

invariant, so that we are allowed to make conformal changes in the Euclidean metric localised around the punctures to obtain a complete metric on $\hat{T} \setminus \{\pm\xi_0\}$. Thus, our strategy is to obtain results concerning the Euclidean metric from known statements about complete metrics.

In [2], Biquard considered the so-called *Poincaré metric*, which is defined as follows. We perform a conformal change on the incomplete metric over the punctured torus localised on small punctured neighbourhoods D_0 of $\pm\xi_0$, so that if $\xi = (r, \theta)$ is a local coordinate on D_0, we have the metric:

$$ds_P^2 \;=\; \frac{d\xi d\bar{\xi}}{|\xi|^2 \log^2 |\xi|^2} \;=\; \frac{dr^2}{r^2 \log^2 r} + \frac{d\theta^2}{4 \log^2 r} \tag{2}$$

We denote the complete metric so obtained by g_P. The Euclidean metric is denoted by g_E. Whenever necessary, we will denote by L^2_E and L^2_P the Sobolev norms in $\Gamma(\Lambda^* V)$ with respect to g_E and g_P, respectively, together with the hermitian metric in V.

Model solutions of (1) in a neighbourhood of the singularities were described by Biquard [3]:

$$\begin{aligned} B &= b\frac{d\xi}{\xi} + b^* \frac{d\bar{\xi}}{\bar{\xi}} \\ \Phi &= \phi_0 \frac{d\xi}{\xi} \end{aligned}$$

where $b, \phi_0 \in \mathfrak{sl}(k)$. Every meromorphic Higgs pair with a simple pole approaches this model close enough to the singularities.

Finally, a Higgs pair (B, Φ) is said to be *admissible* if V has no covariantly constant sections.

3 Construction of doubly-periodic instantons

Our task now is to construct a $SU(2)$ vector bundle over $T \times \mathbb{C}$, with an instanton connection on it, starting from a suitable singular Higgs pair as described in the previous section.

The key feature of Nahm transforms is to try to solve a Dirac equation, and then use its L^2-solutions to form a vector bundle over the dual lattice; see the references in the introduction.

So let $S^+ = \Lambda^0 \oplus \Lambda^{1,1}$ and $S^- = \Lambda^{1,0} \oplus \Lambda^{0,1}$, as vector bundles over $\hat{T}$. The idea is to study the following elliptic operators:

$$\begin{array}{ll} \mathcal{D} : \Gamma(V \otimes S^+) \to \Gamma(V \otimes S^-) & \mathcal{D}^* : \Gamma(V \otimes S^-) \to \Gamma(V \otimes S^+) \\ \mathcal{D} = (\overline{\partial}_B + \Phi) - (\overline{\partial}_B + \Phi)^* & \mathcal{D}^* = (\overline{\partial}_B + \Phi)^* - (\overline{\partial}_B + \Phi) \end{array} \tag{3}$$

where (B, Φ) is a Higgs pair. Note that the operators in (3) are just the Dirac operators coupled to the connection $\widetilde{B}$, obtained by lifting the Higgs pair (B, Φ) to an invariant ASD connection on $\mathbb{R}^4$, as above.

The next step is to prove that the admissibility condition implies the vanishing of the L^2-kernel of $\mathcal{D}$:

Proposition 1. *The Higgs pair (B, Φ) is admissible if and only if L^2_E−ker$\mathcal{D} = \{0\}$.*

Proof: Given a section $s \in L^2_2(V \otimes S^+)$, the Weitzenböck formula with respect to the Euclidean metric on the punctured torus is given by:

$$\begin{aligned} (\overline{\partial}_B^* \overline{\partial}_B + \overline{\partial}_B \overline{\partial}_B^*) s &= \nabla_B^* \nabla_B s + F_B s \;=\; \nabla_B^* \nabla_B s - [\Phi, \Phi^*] s \\ \Rightarrow \quad \nabla_B^* \nabla_B s &= (\overline{\partial}_B^* \overline{\partial}_B + \overline{\partial}_B \overline{\partial}_B^* + \Phi\Phi^* + \Phi^*\Phi) s \\ &= \left\{ (\overline{\partial}_B + \Phi)(\overline{\partial}_B^* + \Phi^*) + (\overline{\partial}_B^* + \Phi^*)(\overline{\partial}_B + \Phi) \right\} s \\ &= \mathcal{D}^* \mathcal{D} s \end{aligned}$$

and integrating by parts, we get:

$$||\mathcal{D}s||^2_{L^2_E} \;=\; ||\nabla_B s||^2_{L^2_E}$$

Thus, if B is admissible, then the L^2_E-kernel of $\mathcal{D}$ must vanish. The converse statement is also clear. □

In other words, the above proposition implies that the L^2_E-cohomology of orders 0 and 2 of the complex:

$$\mathcal{C} \; : \; 0 \to \Lambda^0 V \xrightarrow{\Phi+\overline{\partial}_B} \Lambda^{1,0}V \oplus \Lambda^{0,1}V \xrightarrow{\overline{\partial}_B+\Phi} \Lambda^{1,1}V \to 0 \tag{4}$$

must vanish. On the other hand, since the L^2-norm for 1-forms is conformally invariant, the L^2-cohomology $H^1(\mathcal{C})$ does not depend on the metric itself, only on its conformal class.

Motivated by a result of Biquard (theorem 12.1 in [2]) we will see how one can identify $H^1(\mathcal{C})$ in terms of a certain hypercohomology vector space which we now introduce.

Let $\mathcal{V} \to \hat{T}$ be the extended holomorphic vector bundle mentioned above. Recall that if ξ_0 is not an element of order 2 then the residue of the Higgs field Φ at $\pm\xi_0$ is a $k \times k$ matrix of rank 1. Therefore, if s is a local holomorphic section on a neighbourhood of $\pm\xi_0$, $\Phi(s)$ has at most a simple pole at $\pm\xi_0$ and its residue has the form $(*, 0, \ldots, 0)$ on some suitable trivialisation.

Similarly, if ξ_0 is an element of order 2, $\Phi(s)$ has at most a simple pole at $\pm\xi_0$ and its residue has the form $(*, *, 0, \ldots, 0)$ on some suitable trivialisation.

This local discussion motivates the definition of a sheaf $\mathcal{P}_{\pm\xi_0}$ such that, given an open cover $\{U_\alpha\}$ of $\hat{T}$:

- $\mathcal{P}_{\pm\xi_0}(U_\alpha) = \mathcal{O}_{\hat{T}}(\mathcal{V})(U_\alpha)$, if $\pm\xi_0 \notin U_\alpha$;
- $\mathcal{P}_{\pm\xi_0}(U_\alpha) = \{$meromorphic sections of $U_\alpha \to U_\alpha \times \mathbb{C}^k$ which have at most a simple pole at $\pm\xi_0$ with residue lying either along a 2-dimensional subspace of $\mathbb{C}^k$ if ξ_0 has order 2, or along a 1-dimensional subspace of $\mathbb{C}^k$ otherwise$\}$, if $\pm\xi_0 \in U_\alpha$.

It is easy to see that such $\mathcal{P}_{\pm\xi_0}$ is a coherent sheaf. To simplify notation, we drop the subscript $\pm\xi_0$ out.

Hence, Φ can be regarded as the map of sheaves:

$$\Phi : \mathcal{V} \to \mathcal{P} \otimes K_{\hat{T}} \tag{5}$$

Seen as a two-term complex of sheaves, the map (5) induces an exact sequences of hypercohomology vector spaces:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{H}^0(\hat{T},\Phi) & \to & H^0(\hat{T},\mathcal{V}) & \xrightarrow{\Phi} & H^0(\hat{T},\mathcal{P}\otimes K_{\hat{T}}) & \to \\ & \to & \mathbb{H}^1(\hat{T},\Phi) & \to & H^1(\hat{T},\mathcal{V}) & \xrightarrow{\Phi} & H^1(\hat{T},\mathcal{P}\otimes K_{\hat{T}}) & \to \\ & \to & \mathbb{H}^2(\hat{T},\Phi) & \to & 0 & & & \end{array} \tag{6}$$

It is easy to see that:

$$\begin{aligned} \mathbb{H}^0(\hat{T},\Phi) &= \ker\left\{H^0(\hat{T},\mathcal{V}) \xrightarrow{\Phi} H^0(\hat{T},\mathcal{P}\otimes K_{\hat{T}})\right\} \\ \mathbb{H}^2(\hat{T},\Phi) &= \operatorname{coker}\left\{H^1(\hat{T},\mathcal{V}) \xrightarrow{\Phi} H^1(\hat{T},\mathcal{P}\otimes K_{\hat{T}})\right\} \end{aligned}$$

and admissibility implies that the right-hand sides must vanish: restricted to $\hat{T}\setminus\{\pm\xi_0\}$, a section there would give a section in the kernel of $\mathcal{D}$ (or, equivalently, a class in $H^0(\mathcal{C})$ and $H^1(\mathcal{C})$). Therefore, the dimension of $\mathbb{H}^1(\hat{T},\Phi)$ is equal to $\chi(\mathcal{P}\otimes K_{\hat{T}}) - \chi(\mathcal{V}) = \chi(\mathcal{P}) - \chi(\mathcal{V})$.

To compute this number, note that there is also a natural map $\mathcal{V} \xrightarrow{\iota} \mathcal{P}$ defined as the *local inclusion* of holomorphic local sections (elements of $\mathcal{O}_{\hat{T}}(\mathcal{V})(U_\alpha)$), into the meromorphic ones (elements of $\mathcal{P}(U_\alpha)$). It fits into the following sequence of sheaves:

$$0 \to \mathcal{V} \xrightarrow{\iota} \mathcal{P} \xrightarrow{res_{\xi_0}} \mathcal{R}_{\xi_0} \to 0 \qquad \text{if } \xi_0 \text{ has order 2,} \tag{7}$$

$$0 \to \mathcal{V} \xrightarrow{\iota} \mathcal{P} \xrightarrow{res_{\pm\xi_0}} \mathcal{R}_{\pm\xi_0} \to 0 \qquad \text{otherwise} \tag{8}$$

where $\mathcal{R}_{\xi_0}$ is the skyscraper sheaf supported at ξ_0 and stalk isomorphic to $\mathbb{C}^2$ and $\mathcal{R}_{\pm\xi_0}$ is the skyscraper sheaf supported at $\pm\xi_0$ and stalks isomorphic to $\mathbb{C}$. Since $\chi(\mathcal{R}_{\pm\xi_0}) = \chi(\mathcal{R}_{\xi_0}) = 2$, we conclude that $\mathbb{H}^1(\hat{T},\Phi)$ is a 2-dimensional complex vector space.

Proposition 2. *The hypercohomology induced by the map of sheaves (5) coincides with the L^2_P-cohomology of the complex (4).*

In particular, we have identifications:

$$\mathbb{H}^1(\hat{T},\Phi) \;\equiv\; L^2_P\text{–cohomology } H^1(\mathcal{C}) \;\equiv\; L^2_E\text{–cohomology } H^1(\mathcal{C})$$

Furthermore, note also that the L^2_E-cohomology of 1-forms with respect to the Euclidean metric is a 2-dimensional complex vector space.

Proof: The hypercohomology defined by the map (5) is given by the total cohomology of the double complex:

$$\begin{array}{ccc} \Lambda^0\mathcal{V} & \stackrel{\Phi}{\to} & \Lambda^{1,0}\mathcal{P} \\ \overline{\partial}\downarrow & & \downarrow\overline{\partial} \\ \Lambda^{0,1}\mathcal{V} & \stackrel{\Phi}{\to} & \Lambda^{1,0}\mathcal{P} \end{array}$$

which in turns is just the cohomology of the complex:

$$0 \to \Lambda^0\mathcal{V} \stackrel{\Phi+\overline{\partial}}{\to} \Lambda^{1,0}\mathcal{P} \oplus \Lambda^{0,1}\mathcal{V} \stackrel{\overline{\partial}+\Phi}{\to} \Lambda^{1,0}\mathcal{P} \to 0$$

Now restricting the complex above to the punctured torus $\hat{T} \setminus \{\pm\xi_0\}$, we get:

$$0 \to \Lambda^0 V \stackrel{\Phi+\overline{\partial}_B}{\to} \Lambda^1 V \stackrel{\overline{\partial}_B+\Phi}{\to} \Lambda^2 V \to 0$$

which is, of course, the complex $\mathcal{C}$.

So, let s be a section of $\Lambda^{1,0}\mathcal{P} \oplus \Lambda^{0,1}\mathcal{V}$ defining a class in $\mathbb{H}^1(\hat{T},\Phi)$. Thus, restricting s to $\hat{T} \setminus \{\pm\xi_0\}$ yields a section s_r of $L^2(\Lambda^1 V)$ defining a class in $H^1(\mathcal{C})$.

Such *restriction map* is clearly a well-defined map:

$$\begin{array}{rcl} R : \mathbb{H}^1(\hat{T},\Phi) & \to & H^1(\mathcal{C}) \\ < s > & \to & < s_r > \end{array}$$

We claim that it is also injective. Indeed, suppose that s_r represents the zero class, i.e. there is $t \in L^2_2(\Lambda^0 V)$ such that $s_r = (\overline{\partial}_B + \Phi)t$. However, $L^2_2 \hookrightarrow C^0$ is a bounded inclusion in real dimension 2. Thus, $h(t,t)$ must be bounded at the punctures $\pm\xi_0$, and t must be itself bounded along the kernel of the residues of Φ. On the other hand, the hermitian metric degenerates along the image of the residues of Φ, so t might be singular on this direction. Indeed, $h \sim O(r^{1\pm\alpha})$ in a holomorphic trivialisation, so that $t \sim O(r^{-\frac{1}{2}(1\pm\alpha)})$.

But then the derivatives of t will not be square integrable, contradicting our hypothesis that t belongs to L_2^2. So t must be bounded at $\pm\xi_0$.

This implies that $t \in L_2^2(\Lambda^0\mathcal{V})$ also with respect to the h' metric, so that s_r is indeed the restriction of a section representing the zero class in $\mathbb{H}^1(\hat{T}, \Phi)$.

Finally, to show that R is an isomorphism, it is enough by admissibility to argue that the L^2 index of the complex $\mathcal{C}$ is -2.

It was shown by Biquard (theorem 5.1 in [2]) the laplacian associated to the complex $\mathcal{C}$ is Fredholm when acting between L_P^2 sections. This implies that $\mathcal{D}$ is also Fredholm. Its index can be computed via Gromov-Lawson's relative index theorem, and it coincides with the index of the Dirac operator on $\mathcal{V}$:

$$\text{index}(\mathcal{D}) = \text{index}(\overline{\partial}_B - \overline{\partial}_B^*) = \deg\mathcal{V} = -2$$

as desired $\square$

Constructing the transformed bundle. We are finally in a position to construct a vector bundle with connection over $T \times \mathbb{C}$ out of a Higgs pair (B, Φ). Recall that $\mathcal{J}(\hat{T}) = T$, the *Jacobian* of $\hat{T}$, is defined as the set of flat holomorphic line bundles over $\hat{T}$. Each $z \in T$ corresponds to a flat holomorphic line bundle $L_z \to \hat{T}$. Moreover, T and $\hat{T}$ are isomorphic as elliptic curves.

These line bundles can be given a natural constant connection compatible with the holomorphic structure. This follows from the differential-geometric definition of T:

$$T = \{z \in (\mathbb{R}^2)^* \mid z(\xi) \in \mathbb{Z}, \forall \xi \in \Lambda\}$$

where $\Lambda \subset \mathbb{R}^4$ is the two-dimensional lattice generating $\hat{T}$. Hence each $z \in T$ can be regarded as a constant, real 1-form over $\hat{T}$, so that $\omega_z = i \cdot z$ is a connection on a topologically trivial line bundle $L \to \hat{T}$. Each such connection defines a different holomorphic structure on L, which we denote by L_z.

Conversely, $\hat{T}$ parametrises the set of holomorphic flat line bundles with connection over T. Each point $\xi \in \hat{T}$ corresponds to the line bundle $L_\xi \to T$ with a connection ω_ξ.

Now consider the restrictions $L_z \to \hat{T}\backslash\{\pm\xi_0\}$, with its natural connection ω_z, and form the tensor product $V(z) = V \otimes L_z$. The connection B can be tensored with ω_z to obtain another connection that we denote by B_z.

Let $i : V(z) \to V(z)$ be the identity bundle automorphism and define $\Phi_w = \Phi - w \cdot i$, where w is a complex number. It is easy to see that (B_z, Φ_w) is still an admissible Higgs pair, for all $(z, w) \in T \times \mathbb{C}$.

Therefore, we get the following continuous family of Dirac-type operators:

$$\mathcal{D}_{(z,w)} = (\overline{\partial}_{B_z} + \Phi_w) - (\overline{\partial}_{B_z} + \Phi_w)^* \tag{9}$$

From proposition 1, we have that L^2_E–ker$\mathcal{D}_{(z,w)}$ vanishes for all $(z, w) \in T \times \mathbb{C}$. Since its index remains invariant under this continuous deformation, we conclude that L^2_E–ker$\mathcal{D}^*_{(z,w)}$ has constant dimension equal to 2.

Define a trivial Hilbert bundle $H \to T \times \mathbb{C}$ with fibres given by $L^2(V(z) \otimes S^-)$. It follows that $E_{(z,w)} = \ker\mathcal{D}^*_{(z,w)}$ forms a vector sub-bundle $E \stackrel{i}{\hookrightarrow} H$ of rank 2. Furthermore [7], E is also equipped with an hermitian metric, induced from the L^2 metric on H, and an unitary connection A, defined as follows:

$$\nabla_A = P \circ d \circ i \tag{10}$$

where d means differentiation with respect to (z, w) on the trivial Hilbert bundle (i.e. the trivial product connection) and P is the fibrewise orthogonal projection $P : L^2(V(z) \otimes S^-) \to \ker\mathcal{D}^*_{(z,w)}$. Clearly, A defined on (10) is unitary.

Note also that the hermitian metric in H is actually conformally invariant with respect to the choice of metric in $\hat{T} \setminus \{\pm\xi_0\}$, since the inner product in $L^2(V(z) \otimes S^-)$ is. Therefore, the induced hermitian metric in E is also conformally invariant.

Monad description. The transformed bundle E also admits a monad-type description. More precisely, once a metric is chosen, the family of Dirac operators $\ker\mathcal{D}^*_{(z,w)}$ can be unfolded into the following family of elliptic complexes $\mathcal{C}(z,w)$:

$$0 \to L^2_{2,E}(\Lambda^0 V(z)) \stackrel{\Phi_w+\overline{\partial}_{B_z}}{\longrightarrow} L^2_{1,E}(\Lambda^{1,0}V(z) \oplus \Lambda^{0,1}V(z)) \stackrel{\overline{\partial}_{B_z}+\Phi_w}{\longrightarrow} L^2_E(\Lambda^{1,1}V(z)) \to 0 \tag{11}$$

Admissibility implies that $H^0(\mathcal{C}(z,w))$ and $H^2(\mathcal{C}(z,w))$ must vanish, and $H^1(\mathcal{C}(z,w))$ coincides with $L^2_E{-}\ker\mathcal{D}^*_{(z,w)}$. As (z,w) sweeps out $T \times \mathbb{C}$, $H^1(\mathcal{C}(z,w))$ forms a rank 2 holomorphic vector bundle with a natural hermitian metric and a compatible unitary connection A, equivalent to the ones defined as above; see [7].

3.1 Anti-self-duality and curvature decay

The next proposition fulfills the first goal of this paper, i.e. to show that the connection A defined above is in fact a finite-energy anti-self-dual instanton on the rank 2 bundle $E \to T \times \mathbb{C}$. We say $f \sim O(|w|^n)$ if the complex function $f : \mathbb{C} \to \mathbb{C}$ satisfies:

$$\lim_{|w|\to\infty} \frac{|f(w)|}{|w|^n} < \infty \tag{12}$$

Theorem 3. *The transformed connection A is anti-self-dual with respect to the Euclidean metric. Furthermore, its curvature satisfies $|F_A| \sim O(|w|^{-2})$.*

Proof: Since A is an unitary connection, we only have to verify that the component of F_A along the Kähler class κ of $T \times \mathbb{C}$ vanishes.

Let $\{\psi_1, \psi_2\}$ be a local holomorphic frame for E, orthonormal with respect to the hermitian metric induced from H. Fix some $(z,w) \in T \times \mathbb{C}$ so that, as a section of $\mathcal{V}(z) \otimes S^- \to \hat{T}$, we have $\psi_i = \psi_i(\xi; z, w) \in \ker\mathcal{D}^*_{(z,w)}$.

In this trivialisation, the matrix elements of the curvature F_A can then be written as follows:

$$(F_A)_{ij} \;=\; \langle \psi_j, \nabla_A \nabla_A \psi_i \rangle \;=\; \langle \psi_j, d \circ P \circ d\psi_i \rangle \;=$$

$$= \quad \langle \mathcal{D}^*_{(z,w)}(d\psi_j), G_{(z,w)}\mathcal{D}^*_{(z,w)}(d\psi_j)\rangle \tag{13}$$

where the inner product is taken in $L^2(V(z)\otimes S^-)$, integrating out the ξ coordinate; the finiteness of the integral is guaranteed by the fact that $\psi_j \in L^2_1(V(z)\otimes S^-)$. Note also that the inner product is conformally invariant with respect to the choice of metric on $\hat{T}\setminus\{\pm\xi_0\}$. Hence, the expression for the curvature above is the same for both the Euclidean and Poincaré metrics.

Moreover, $G_{(z,w)}$ is the Green's operator for $\mathcal{D}^*_{(z,w)}\mathcal{D}_{(z,w)}$. Note that:

$$[\mathcal{D}^*_{(z,w)}, d]\psi_i = \Omega' \cdot \psi_i$$

where $\Omega' = (idz_1 + dw_1)\wedge d\xi_1 + (idz_2 + dw_2)\wedge d\xi_2$ and "$\cdot$" denotes Clifford multiplication. So,

$$\kappa\llcorner(F_A)_{ij} \quad = \quad \langle \psi_j, \underbrace{\kappa\llcorner(\Omega'\wedge\Omega')}_{=0}\cdot G_{(z,w)}\psi_i\rangle \;=\; 0 \tag{14}$$

and this proves the first statement.

It is easy to see from (14) that the asymptotic behaviour of $|(F_A)_{ij}|$ depends only on the behaviour of the operator norm $||G_{(z,w)}||$ for large $|w|$.

We can estimate $||G_{(z,w)}||$ by looking for a lower bound for the eigenvalues of the associated laplacian acting on $V\otimes S^-$:

$$\mathcal{D}_{(z,w)}\mathcal{D}^*_{(z,w)} \;=\; \mathcal{D}_z\mathcal{D}^*_z - w\phi^* - \overline{w}\phi + |w|^2 \tag{15}$$

where $\mathcal{D}_z = \mathcal{D}_{(z,w=0)}$ and $\Phi = \phi d\xi$, with $\phi \in \mathrm{End}V$; ϕ^* denotes the adjoint (conjugate transpose) endomorphism.

In other words, we want to find a lower bound for the following expression:

$$\begin{gathered} |\langle(\mathcal{D}_z\mathcal{D}^*_z + |w|^2)s, s\rangle - \langle(w\phi^* + \overline{w}\phi)s, s\rangle| \geq \\ \geq |\; \langle(\mathcal{D}_z\mathcal{D}^*_z + |w|^2)s, s\rangle \;-\; |\langle(w\phi^* + \overline{w}\phi)s, s\rangle| \;| \end{gathered} \tag{16}$$

for $s \in L^2(V\otimes S^-)$ of unit norm.

For the first term in the second line, it is easy to see that:

$$|\langle(\mathcal{D}_z\mathcal{D}_z^* + |w|^2)s, s\rangle| \;=\; ||\mathcal{D}_z^* s||^2 + |w|^2 \cdot ||s||^2 \;=\; c_1 + |w|^2 \qquad (17)$$

for some non-zero constant $c_1 = ||\mathcal{D}_z^*||^2$ depending only on $z \in T$.

The second term in (16) is more problematic; first note that:

$$|\langle(w\phi^* + \overline{w}\phi)s, s\rangle| \;\leq\; |w| \cdot (|\langle\phi(s), s\rangle| + |\langle\phi^*(s), s\rangle|)$$

In a small neighbourhood D_0 of each singularity $\pm\xi_0$, we have:

$$\begin{aligned} \langle\phi(s), s\rangle_{L^2(D_0)} &= \int_{D_0} \langle\frac{\phi_0(s)}{\xi}, s\rangle r dr d\theta + \begin{pmatrix} \text{regular} \\ \text{terms} \end{pmatrix} \\ &\sim \int_{D_0} \frac{|\phi_0|}{r} \cdot |s|^2 r dr d\theta + \begin{pmatrix} \text{regular} \\ \text{terms} \end{pmatrix} \end{aligned}$$

Let $1 < p < 2$; using Hölder inequality, we obtain:

$$\begin{aligned} \int_{D_0} \frac{|\phi_0|}{\xi} \cdot |s|^2 &\leq \left\{\int_{D_0} \left(\frac{|\phi_0|}{r}\right)^p r dr d\theta\right\}^{1/p} \left\{\int_{D_0} |s|^{2q}\right\}^{1/q} \\ &\leq c \cdot ||s||^2_{L^{2q}} \end{aligned}$$

where $q = \frac{p}{p-1}$, and for some real constant c depending only on ϕ_0 and on the choice of p.

Since $2q > 4$, the Sobolev embedding theorem tells us that $L_1^2 \hookrightarrow L^{2q}$ is a bounded inclusion (in real dimension 2). In other words, there is a constant C depending only on q such that $||s||_{L^{2q}} \leq C \cdot ||s||_{L_1^2}$. Thus, arguing similarly for the $\langle\phi^*(s), s\rangle$ term, we conclude that:

$$|\langle(w\phi^* + \overline{w}\phi)s, s\rangle| \;\leq\; c_2 \cdot |w|$$

where c_2 is a real constant depending neither on z nor on w, but only on the Higgs field itself and on the choice of p.

Putting everything together, we have:

$$\left|\langle(\mathcal{D}_z\mathcal{D}_z^* - w\phi^* - \overline{w}\phi + |w|^2)s, s\rangle\right| \geq \left||w|^2 - c_2|w| + c_1\right|$$

so that

$$\lim_{|w|\to\infty} |w|^2 \cdot ||G_{(z,w)}|| < 1$$

and the statement follows. $\square$

Remark 1: Note in particular that $F_A \in L^2(\Lambda^2 \otimes E)$ with respect to the Euclidean metric on $T \times \mathbb{C}$, coming from the quotient $(\mathbb{R}^4)^*/\Lambda^*$. This concludes our first task.

Remark 2: It is also not difficult to see that gauge equivalent Higgs pairs (B, Φ) and (B', Φ') will produce gauge equivalent instantons A and A'. The dependence of A on the Higgs pair (B, Φ) is contained on the L^2-projection operator P, that is on the two linearly independent solutions of $\mathcal{D}^*_{(z,w)}\psi = 0$. Gauge equivalence of (B, Φ) and (B', Φ') gives an automorphism of the transformed bundle E, in other words, a gauge equivalence between A and A'.

Remark 3: The instanton connection A induces a holomorphic structure $\overline{\partial}_A$ on the the transformed bundle $E \to T \times \mathbb{C}$.

In order to further understand the asymptotic behaviour of the transformed connection, we must now pass to an equivalent holomorphic description of the above transform.

4 Holomorphic version and extensibility

Motivated by curvature decay established above, one can expect to find a holomorphic vector bundle $\mathcal{E} \to T \times \mathbb{P}^1$ which extends $(E, \overline{\partial}_A)$. The idea is to find a suitable perturbation of the Higgs field Φ for which $w = \infty$ makes sense.

As above, the torus parameter $z \in T$ simply twists the holomorphic bundle $\mathcal{V} \to \hat{T}$. We denote:

$$\mathcal{V}(z) = \mathcal{V} \otimes L_z \qquad \mathcal{P}(z) = \mathcal{P} \otimes L_z \tag{18}$$

Since $\Phi \in H^0(\hat{T}, \mathrm{Hom}(\mathcal{V}, \mathcal{P}) \otimes K_{\hat{T}})$, tensoring both sides of (5) by the line bundle L_z does not alter the sheaf homomorphism Φ, so we have the family of maps:

$$\Phi : \mathcal{V}(z) \to \mathcal{P}(z) \otimes K_{\hat{T}}$$

parametrised by $z \in T$.

To define the perturbation Φ_w, recall that, regarding $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, we can fix two holomorphic sections $s_0, s_\infty \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ such that s_0 vanishes at $0 \in \mathbb{C}$ and s_∞ vanishes at the point added at infinity. In homogeneous coordinates $\{(w_1, w_2) \in \mathbb{C}^2 | w_2 \neq 0\}$ and $\{(w_1, w_2) \in \mathbb{C}^2 | w_1 \neq 0\}$, we have that, respectively ($w = w_1/w_2$):

$$\begin{array}{ll} s_0(w) = w & \qquad s_0(w) = 1 \\ s_\infty(w) = 1 & \qquad s_\infty(w) = \dfrac{1}{w} \end{array}$$

Consider now the map of sheaves parametrised by pairs $(z, w) \in T \times \mathbb{P}^1$:

$$\begin{gathered} \Phi_w : \mathcal{V}(z) \to \mathcal{P}(z) \otimes K_{\hat{T}} \\ \Phi_w = s_\infty(w) \cdot \Phi - s_0(w) \cdot \iota \cdot d\xi \end{gathered} \tag{19}$$

Clearly, on $\mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}$ this is just $\Phi_w = \Phi - w \cdot \iota$, the same perturbation we defined before. Moreover, if $w = \infty$, then $\Phi_\infty = \iota \cdot d\xi$

The hypercohomology vector spaces $\mathbb{H}^0(\hat{T}, \Phi_w)$ and $\mathbb{H}^2(\hat{T}, \Phi_w)$ of the two-term complex (19) must vanish by admissibility. On the other hand, $\mathbb{H}^1(\hat{T}, \Phi_w)$ also makes sense for $\infty \in \mathbb{P}^1$, and we can define a $SU(2)$ holomorphic vector bundle $\mathcal{E} \to T \times \mathbb{P}^1$ with fibres given by $\mathcal{E}_{(z,w)} = \mathbb{H}^1(\hat{T}, \Phi_w)$. Moreover, $\mathcal{E}$ is actually a *holomorphic extension* of $(E, \overline{\partial}_A)$, in the sense that, holomorphically:

$$\mathcal{E}|_{T \times (\mathbb{P}^1 \setminus \{\infty\})} \simeq (E, \overline{\partial}_A) \tag{20}$$

Equivalently, $\mathcal{E}$ can be seen as the hermitian holomorphic vector bundle induced by the monad

$$0 \to \Lambda^0 \mathcal{V} \stackrel{\Phi + \overline{\partial}}{\to} \Lambda^{1,0} \mathcal{P} \oplus \Lambda^{0,1} \mathcal{V} \stackrel{\overline{\partial} + \Phi}{\to} \Lambda^{1,0} \mathcal{P} \to 0 \tag{21}$$

Consider the metric H' induced from the monad (21) above, while H is induced from the monad (11). Now, H is bounded above by H' because the hermitian metric h on the bundle V in (11) is bounded above by the metric h' on the bundle $\mathcal{V}$ in (21).

We now show that the position of the singularities of the Higgs pair determines the holomorphic type of the restriction of the extended transformed bundle over the added divisor at infinity. First, recall that there is an unique line bundle $\mathbf{P} \to T \times \hat{T}$, the so-called *Poincaré line bundle*, satisfying:

$$\mathbf{P}|_{T\times\{\xi\}} \simeq L_\xi \qquad \mathbf{P}|_{\{z\}\times\hat{T}} \simeq L_{-z}$$

It can be constructed as follows. Identifying T and $\hat{T}$ as before, let Δ be the diagonal inside $T\times\hat{T}$, and consider the divisor $D = \Delta - T\times\hat{e} - e\times\hat{T}$. Then $\mathbf{P} = \mathcal{O}_{T\times\hat{T}}(D)$; it is easy to see that the sheaf so defined restricts as wanted.

Note that although the two restrictions above are flat line bundles over T and $\hat{T}$ respectively, the Poincaré bundle itself is not topologically trivial; in fact, $c_1(\mathbf{P}) \in H^1(T)\otimes H^1(\hat{T}) \subset H^2(T\times\hat{T})$. More precisely, the unitary connection and its corresponding curvature are given by:

$$\omega(z,\xi) = i\pi\cdot\sum_{\mu=1}^{2}(\xi_\mu dz_\mu - z_\mu d\xi_\mu) \ \text{ and } \ \Omega(z,\xi) = 2i\pi\cdot\sum_{\mu=1}^{2} d\xi_\mu\wedge dz_\mu$$

Restricting to each $T\times\{\xi\}$, the line bundles $L_\xi \to T$ are given flat connections $\omega_\xi = i\pi\cdot\sum_{\mu=1}^2 \xi_\mu dz_\mu$, with constant coefficients. Similarly, the line bundles $L_z \to \hat{T}$ are given the flat connections $\omega_z = -i\pi\cdot\sum_{\mu=1}^2 z_\mu d\xi_\mu$ as described in the previous section. Finally, note that:

$$c_1(\mathbf{P}) = \frac{i}{2\pi}\Omega \ \ \Rightarrow \ \ c_1(\mathbf{P})^2 = -2\cdot t\wedge\hat{t}$$

where t and $\hat{t}$ are the generators of $H^2(T)$ and $H^2(\hat{T})$, respectively.

Lemma 4. $\mathcal{E}|_{T_\infty} \equiv L_{\xi_0}\oplus L_{-\xi_0}$

Proof: Substituting $w=\infty\in\mathbb{P}^1$, we get from (19) that $\Phi_\infty = \iota\cdot d\xi$. Therefore, the induced hypercohomology sequence (23) coincides with the

long exact sequence of cohomology induced by the sheaf sequences (7) and (8), which is given by:

$$\begin{aligned} 0 \;\to\; & H^0(\hat{T},\mathcal{V}(z)) \overset{\Phi_\infty}{\to} H^0(\hat{T},\mathcal{P}(z)\otimes K_{\hat{T}}) \to H^0(\hat{T},\mathcal{R}_{\pm\xi_0}(z)) \to \\ \to\; & H^1(\hat{T},\mathcal{V}(z)) \overset{\Phi_\infty}{\to} H^1(\hat{T},\mathcal{P}(z)\otimes K_{\hat{T}}) \to 0 \end{aligned} \tag{22}$$

Hence, $\mathbb{H}^1(\hat{T},(z,\infty)) = H^0(\hat{T},\mathcal{R}_{\pm\xi_0}(z))$. The right hand side is canonically identified with $(L_z)_{\xi_0}\oplus(L_z)_{-\xi_0}$, where by $(L_z)_{\xi_0}$ we mean the fibre of $L_z\to\hat{T}$ over the point $\xi_0\in\hat{T}$.

On the other hand, $(L_z)_{\xi_0} = \mathbf{P}_{(z,\xi_0)} = (L_{\xi_0})_z$, where $\mathbf{P}\to T\times\hat{T}$ is the Poincaré line bundle. Thus, the bundle over T_∞ with fibres given by $H^0(\hat{T},\mathcal{R}_{\pm\xi_0}(z))$ is isomorphic to $L_{\xi_0}\oplus L_{-\xi_0}$, as we wished to prove. □

The topological type of $\mathcal{E}$ is also fixed from the initial data: the rank of the bundle V is translated into the second Chern class of the extended transformed bundle $\mathcal{E}$. In the next lemma, we denote the generator of $H^2(\mathbb{P}^1,\mathbb{Z})$ by p.

Lemma 5. *$ch(\mathcal{E}) = 2 - k\cdot t\wedge p$.*

Proof: The exact sequence:

$$\begin{aligned} 0 \;\to\; & H^0(\hat{T},\mathcal{V}(z)) \overset{\Phi_w}{\to} H^0(\hat{T},\mathcal{P}(z)\otimes K_{\hat{T}}) \to \mathbb{H}^1(\hat{T},(z,w)) \to \\ \to\; & H^1(\hat{T},\mathcal{V}(z)) \overset{\Phi_w}{\to} H^1(\hat{T},\mathcal{P}(z)\otimes K_{\hat{T}}) \to 0 \end{aligned} \tag{23}$$

induces a sequence of coherent sheaves over $T\times\mathbb{C}$, with stalks over (z,w) given by the above cohomology groups:

$$\begin{aligned} 0 \;\to\; & \mathcal{H}^0(\hat{T},\mathcal{V}(z)) \overset{\Phi_w}{\to} \mathcal{H}^0(\hat{T},\mathcal{P}(z)\otimes K_{\hat{T}}) \to \check{\mathcal{E}} \to \\ \to\; & \mathcal{H}^1(\hat{T},\mathcal{V}(z)) \overset{\Phi_w}{\to} \mathcal{H}^1(\hat{T},\mathcal{P}(z)\otimes K_{\hat{T}}) \to 0 \end{aligned} \tag{24}$$

In this way, the Chern character of $\check{\mathcal{E}}$ will then be given by the alternating sum of the Chern characters of these sheaves, which can be computed via the usual Grothendieck-Riemann-Roch for families.

Consider the bundle $\mathbf{G}_1 \to T\times\mathbb{P}^1\times\hat{T}$ given by $\mathbf{G}_1 = p_3^*\mathcal{V}\otimes p_{13}^*\mathbf{P}$. Clearly, $\mathbf{G}_1|_{(z,w)\times\hat{T}} = \mathcal{V}(z)$, so that:

$$ch(\mathcal{H}^0(\hat{T},\mathcal{V}(z))) - ch(\mathcal{H}^1(\hat{T},\mathcal{V}(z))) = ch(\mathbf{G}_1)td(\hat{T})/[\hat{T}] \qquad (25)$$

Now consider the sheaf: $\mathbf{G}_2 = p_3^*\mathcal{P}\otimes p_{13}^*\mathbf{P}\otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$. The twisting by $\mathcal{O}_{\mathbb{P}^1}(1)$ accounts for the multiplication by the section $s_0 \in H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1))$ contained in Φ_w. As above, $\mathbf{G}_1|_{(z,w)\times\hat{T}} = \mathcal{P}(z)$, and we have:

$$ch(\mathcal{H}^0(\hat{T},\mathcal{P}(z)\otimes K_{\hat{T}})) - ch(\mathcal{H}^1(\hat{T},\mathcal{P}(z)\otimes K_{\hat{T}})) = ch(\mathbf{G}_2)td(\hat{T})/[\hat{T}] \quad (26)$$

Therefore:

$$\begin{aligned}
ch(\mathcal{E}) &= (26)-(25) = \\
&= \left(c_1(\mathcal{P}) - c_1(\mathcal{V}) + c_1(\mathcal{P})\wedge p + \frac{k}{2}c_1(\mathbf{P})^2\wedge p\right)/[\hat{T}] = \\
&= \chi(\mathcal{P}) - \deg\mathcal{V} + \chi(\mathcal{P})\cdot p - k\cdot t\wedge p = 2 - k\cdot t\wedge p
\end{aligned}$$

as desired. □

Finally, we argue that the determinant bundle of $\mathcal{E}$ is trivial, so that A is indeed a $SU(2)$ instanton. Note that $\det\mathcal{E}$ is a line bundle with vanishing first Chern class, so it must be the pull back of a flat line bundle $L_\xi \to T$. But $\det\mathcal{E}|_{T_\infty} = \underline{\mathbb{C}}$, hence $\det\mathcal{E}$ must be holomorphically trivial, as desired.

We call $\xi_0 \in \mathcal{J}(T)$ the *asymptotic state* associated to the doubly-periodic instanton connection A, and the integer k its *instanton number*. The Nahm transform constructed above guarantees the existence of doubly-periodic instantons of any given charge and asymptotic state.

4.1 Extensible doubly-periodic instanton connections

Motivated by the properties established above, we say that a doubly-periodic instanton connection A on a bundle $E \to T\times\mathbb{C}$ is *extensible* if the following hypothesis hold:

1. $|F_A| \sim O(|w|^{-2})$;

2. there is a holomorphic vector bundle $\mathcal{E} \to T \times \mathbb{P}^1$ with trivial determinant such that $\mathcal{E}|_{T\times(\mathbb{P}^1\setminus\{\infty\})} \simeq (E, \overline{\partial}_A)$, where $\overline{\partial}_A$ is the holomorphic structure on E induced by the instanton connection A;

This definition will be our starting point in [14], where we shall present the Nahm transform of doubly-periodic instantons, i.e. the inverse of the construction shown here.

5 Conclusion

In this paper we have shown how finite energy, doubly-periodic instantons can be produced by performing a Nahm transform on certain singular Higgs pairs. The rank of the Higgs bundle is translated into the instanton number; the number of singularities of the Higgs field (i.e. the degree of the holomorphic Higgs bundle $\mathcal{V}$) gives the rank of the transformed instanton, and its positions determine how the instanton connection "splits at infinity". Indeed, it is easy to generalise the above construction by allowing more than two singularities on the original Higgs field, so that higher rank doubly-periodic instantons are obtained; see [14].

Moreover, one would also like to understand how the parabolic parameters (α, ϵ) are translated into the doubly-periodic instantons produced via the Nahm transform as above. On general grounds, we expect these parameters to be translated into more detailed information on the asymptotic behaviour of A.

From the more analytical point of view, it is also interesting to ask if the curvature decay (proposition 3) is enough to ensure extensibility. More precisely, one can expect to be able to prove the following result:

Conjecture 6. *If A is anti-self-dual and $|F_A| \sim O(|w|^{-2})$, then there is a*

holomorphic vector bundle $\mathcal{E} \to T \times \mathbb{P}^1$ *such that*

$$\mathcal{E}|_{T\times(\mathbb{P}^1\setminus\{\infty\})} \simeq (E, \overline{\partial}_A)$$

In other words, A is extensible.

Such conjecture motivates other questions:

- Do all anti-self-dual connections on $E \to T \times \mathbb{C}$ with finite energy with respect to the Euclidean metric satisfy $|F_A| \sim O(|w|^{-2})$?

- Does the converse holds, i.e. if A is extensible then $|F_A| \sim O(|w|^{-2})$? If not, what are the necessary and sufficient analytical conditions for extensibility (in terms of the Euclidean metric)?

- Given a holomorphic bundle $\mathcal{E} \to T \times \mathbb{P}^1$, is there a connection A on $\mathcal{E}|_{T\times(\mathbb{P}^1\setminus\{\infty\})}$ such that A is anti-self-dual and $|F_A| \sim O(|w|^{-2})$ with respect to the Euclidean metric?

We hope to address these issues in a future paper [4].

References

[1] Atiyah, M.; Drinfel'd, V.; Hitchin, N.; Manin, Y.: Construction of instantons. Phys. Lett. A **65**, 185-187 (1978).

[2] Biquard, O.: Fibrés de Higgs et connexions intégrables: le cas logarithmique (diviseur lisse). Ann. Scient. Éc. Norm. Sup. (4) **30**, 41-96 (1997).

[3] Biquard, O.: Sur les équations de Nahm et la structure de Poisson es algébres de Lie semi-simple complexes. Math. Ann. **304**, 253-276 (1996).

[4] Biquard, O.; Jardim, M.: Asymptotic behaviour and the moduli space of doubly-periodic instantons. Preprint math.DG/0005054.

[5] Braam, P.; van Baal, P.; Nahm's transform for instantons. Commun. Math. Phys. **122**, p.267-280 (1989).

[6] Donaldson, S.: Nahm's equations and classification of monopoles. Commun. Math Phys. **96**, 387-207 (1984).

[7] Donaldson, S.; Kronheimer, P.: Geometry of four-manifolds. Oxford: Clarendon Press 1990.

[8] Garland, H.; Murray, M.: Kac-Moody monopoles and periodic instantons. Commun. Math. Phys. **120**, 335-351 (1988).

[9] Gromov, M.; Lawson, H.: Positive scalar curvature and the index of the Dirac operator on complete Riemannian manifolds. Inst. des Hautes Études Scientifiques Publ. Math. **58**, 295-408 (1983).

[10] Hitchin, N.: Construction of monopoles. Commun. Math. Phys. **89**, 145-190 (1983).

[11] Hitchin, N.: The self-duality equations on a Riemann surface. Proc. London Math. Soc. **55**, 59-126 (1987).

[12] Hurtubise, J.; Murray, M.: On the construction of monopoles for the classical groups. Commun. Math. Phys. **122**, 35-89 (1989).

[13] Jardim, M.: Nahm transform for doubly-periodic instantons. Ph.D. thesis, Oxford (1999). Preprint math.DG/9912028.

[14] Jardim, M.: Nahm transform for doubly-periodic instantons. Preprint math.DG/9910120.

[15] Kapustin, A.; Sethi, S.: Higgs branch of impurity theories. Adv. Theor. Math. Phys. **2**, 571-592 (1998).

[16] Konno, H.: Construction of the moduli space of stable parabolic Higgs bundles on a Riemann surface. J. Math. Soc. Japan **45**, 253-276 (1993).

[17] Kovalev, A.: The geometry of dimensionally reduced anti-self-duality equations. Ph.D. thesis, Oxford (1995).

[18] Nahm, W.: Self-dual monopoles and calorons. In: Denardo, G.; Ghirardi, G.; Weber, T. (eds.) Group theoretical methods in physics. Proceedings, Trieste 1980, pp. 189-200. Berlin, New York: Springer-Verlag 1984.

[19] H. Nakajima. *Monopoles and Nahm's equations.* In: Mabuchi, T.; Mukai, S. (eds.) Einstein metrics and Yang-Mills connections. Proceedings, Sanda 1990, pp. 193-211. New York: Marcel Dekker 1993.

[20] Nakajima, H.: Hyperkähler structures on the moduli spaces of parabolic Higgs bundles on Riemann surfaces. In: Maruyama, M. (ed.) Moduli of vector bundles. Proceedings, Sanda 1994; Kyoto 1994, pp. 199-208. New York: Marcel Dekker 1996.

[21] Simpson C.:Harmonic bundles on noncompact curves. J. of Am. Math. Soc. **3**, 713-770 (1990).

[22] Witten, E.: Talk presented at the meeting "Interfaces in Mathematics", in honor of the 70th birthday of Michael Atiyah, London, April 1999.